CONTENTS

November 2024

Discrete Structure

Date ______
Page ______

Group A

1. Why do we need quantifier?

=> We need quantifiers to express the extend to which a predicate is true over a range of elements (domain of discourse)
There are two kinds of quantifiers:
i) Universal Quantifier ($\forall$)
ii) Existential Quantifier ($\exists$)

2. Define pseudorandom integer.

=> It is a number generated by a deterministic algorithm that produces a sequence ~~predic~~ of numbers appearing random but is actually predictable and repeatable.

(3) Give an example of recursively defined set.

⇒ The set of natural numbers N can be defined recursively.

Basis step: $0 \in N$

Recursive step: If $n \in N$, then $n+1 \in N$

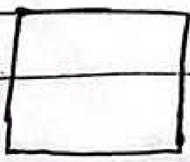
(4) When do you use sum rule?

⇒ Sum rule is used when one task can be done in n_1 ways and another independent task can be done in n_2 ways, then choosing either task can be done in $m+n$ ways.

$|m \cup n| = |m| + |n|$ as long as m and n are disjoint.

(5) Define simple graph.

⇒ A simple graph is an undirected graph that has no loops and multiedges between any two different vertices.

eg: 

(6) Define logic.

⇒ Logic is the study of the principles of valid reasoning that provides a formal framework for determining the truth of statements and arguments.

eg: If it rains, then the ground gets wet
$p \to q$

(7) What does prefix code mean?

⇒ Prefix code is the computer friendly code in which operator precedes its operands.

eg: +ab
-+ abc

(8) How do you define hypothesis in strong induction.

⇒ In strong induction, the inductive hypothesis is the assumption that the statement P(k) is true for all integers from base case up to an arbitrary integer k, and then we use this to prove that P(k+1) is true.

(9) Differentiate between homogeneous and non homogeneous recurrence relation.

⇒ A homogeneous recurrence relation has the form,

$a_n = c_1 a_{n-1} + c_2 a_{n-2} + c_3 a_{n-3} + \text{---} + c_k a_{n-k}$

A non homogeneous recurrence relation has an additional function of n that is not the multiple of the sequence itself.

eg: $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \text{---} \cdot c_k a_{n-k} + f(n)$

(10) Define binary search tree.

⇒ Binary Search Tree (BST) is a binary tree in which every node/vertex has at most 2 childs.

Group B

(1) Compute the value of 5 mod 6, −5 mod 6 and 2 mod 2

We know,

a mod n = remainder when a is divided b

i) 5 mod 6

$5 = 6 \times 0 + \underline{5}$

Hence 5 mod 6 = 5

ii) −5 mod 6

$-5 = 6(-1) + 1$

Hence, −5 mod 6 = 1

iii) 2 mod 2

$2 = 2 \times 1 + 0$

Hence 2 mod 2 = 0

2) Use the extended Euclidean algorithm to find the GCD of 16 and 28.

→ Soln,

q	a	b	r	s_1	s_2	s	t_1	t_2	t
1	28	16	12	1	0	1	0	1	-1
1	16	12	4	0	1	-1	1	-1	2
3	12	4	0	1	-1	4	-1	2	-5
	$\boxed{4}$	0		$\boxed{-1}$	4		$\boxed{2}$		

Initially, $s_1 = 1$ $s_2 = 0$ | $s = s_1 - q s_2$

$t_1 = 0$ $t_2 = 1$ | $t = t_1 - q t_2$

We have,

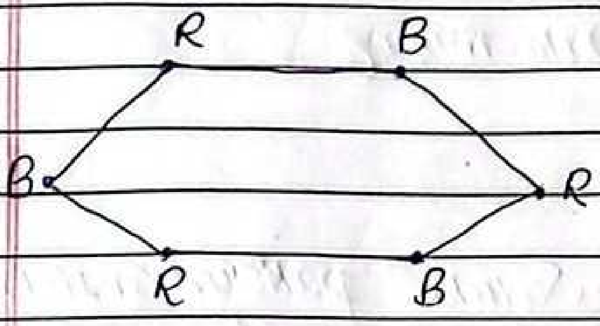
$$s \cdot a + t \, b = \gcd(a, b)$$

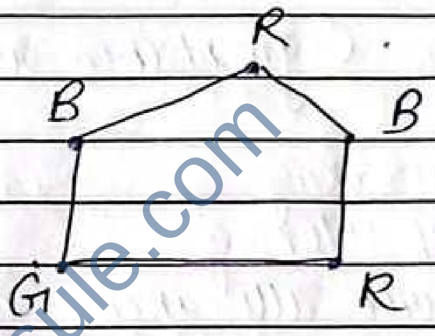
$$= -1 \cdot 28 + 2 \cdot 16 = 4$$

Hence, $\gcd(16, 28) = 4$,,

3) ## Chromatic number

The least no. of colors required for coloring of a graph G is called chromatic number. It is denoted by $\chi(G)$
If $\chi(G) = K$, then graph is called K-chromatic.



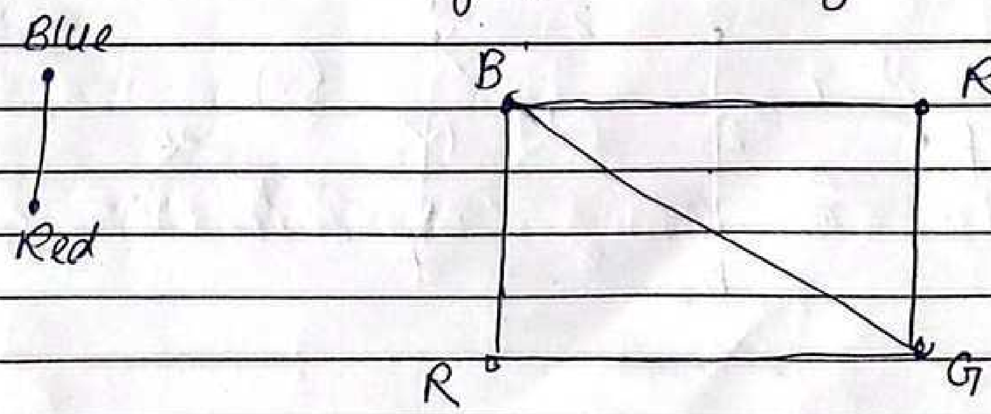


$\chi(G) = 2$, 2-chromatic $\chi(G) = 3$, 3-chromatic

Graph coloring

Painting all the vertices of graphs with colors such that no two adjacent vertices have the same color is called graph coloring.

eg:



Application of Graph coloring

1) Scheduling Problems

⇒ Graph coloring is widely used in scheduling tasks where conflicts must be avoided.
eg: course scheduling, exam scheduling

II) Image Segmentation and Data mining

III) Networking

IV) Map coloring

V) Resource allocation, routing, Sudoku, pattern matching and clustering.

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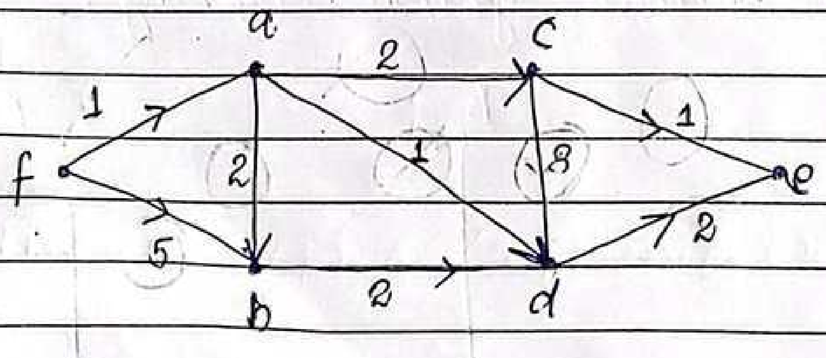
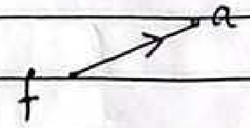
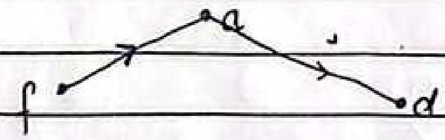
(14) Find the minimum spanning tree using Prim's algorithm.



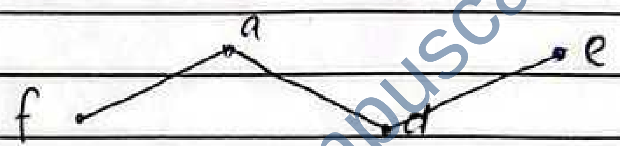
① Select a vertex 'f' as starting point, edge set:
(f, a) = 1, (f, b) = 5



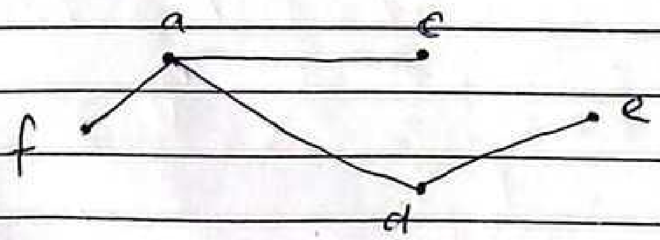
② edge set: (f, b) = 5, (a, b) = 2, (a, c) = 2, (a, d) = 1



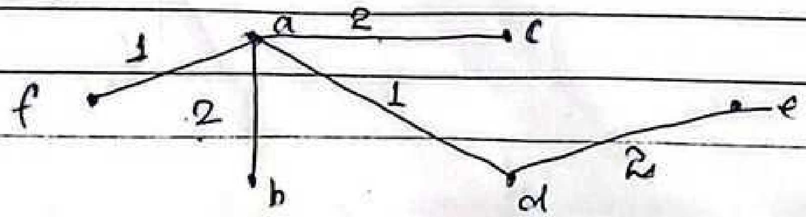
③ edge set: (f, b) = 5, (a, b) = 2, (a, c) = 2, (d, e) = 2



④ edge set: (f, b) = 5, (a, b) = 2, (a, c) = 2



⑤ ~~edge set: (f, b) = 5, (a, b) = 2, (c, d) = 3, (c, e) = 1~~



= 8 //

15) Create a BST from 6, 5, 8, 10, 2, 100 and traverse in post order.

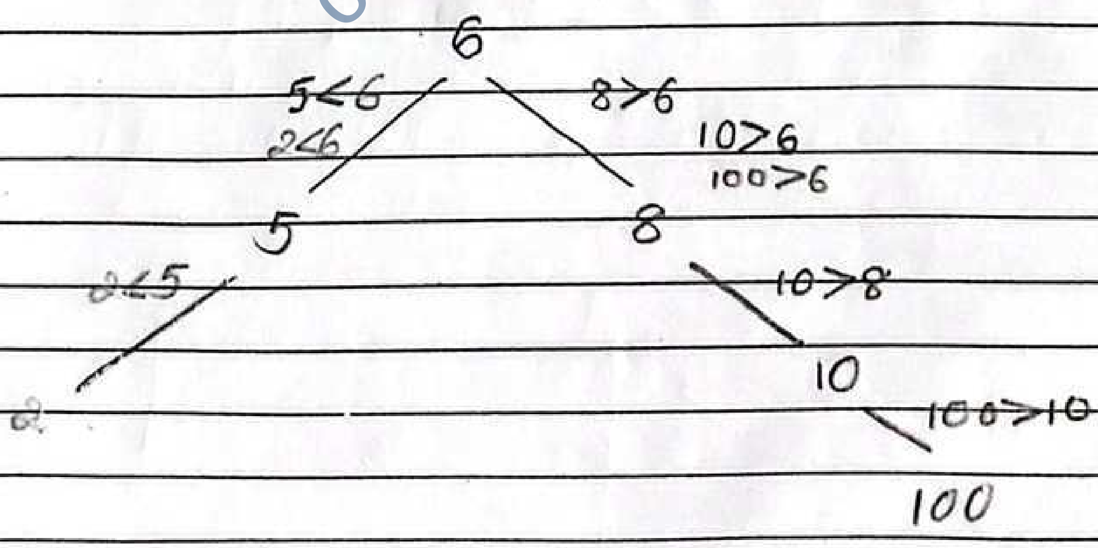
BST (Binary Search Tree)

=) It is a binary tree in which every node/vertex has at most 2 childs, and keeps its data in sorted order of pattern.

Rules:

- Each node has at most two children - a left child and a right child.

- The left child's value is always smaller than parent's value
- The right child's value is always larger than parent's value



Postorder Traversal (left, right, root)

Step 1   5   8   6

2   10

100

Step 2   2   5   10   8   6

100

Step 3   2 , 5 , 100 , 10 , 8 , 6

Group 'C'

17) Using mathematical induction prove that $n! > 2n$ for $n \geq 4$

Base case: $P(4)$ is true, $4! = 24 \geq 8$.

Inductive step: We assume that $P(k)$ is true for an arbitrary non negative k. i.e
$k! \geq 2k$ is true

We must show that $P(k+1)$ is true.
$(k+1)! \geq 2(k+1)$

We have,

$$(k+1)! = (k+1)\,k!$$
$$\geq (k+1) \cdot 2k$$
$$= 2k(k+1)$$
$$\geq 2(k+1) \quad [\because \text{ Since } k \geq 4]$$

Hence by induction, $n! \geq 2n$ for all integers $n \geq 4$.

18) Using proof by contradiction, show that the sum of odd integers is even.

→ Soln

To prove the statement by contradiction, we assume that, the sum of odd integers is odd.

Let the two odd integers be,

$a = 2m + 1$

$b = 2n + 1$ where m and n are integers.

Computing the sum

$a + b = 2m + 1 + 2n + 1$

$= 2m + 2n + 2$

$= 2(m + n + 1)$

$= 2k$ ($m+n+1$, can be written as k)

which is even.

We assumed the sum of two odd numbers was odd, but we found it is even. Hence, by contradiction,

the sum of odd integers must be even.

19) Solve the recurrence relation, $a_n = 3a_{n-1} - 2a_{n-2}$, with initial conditions $a_0 = 1$, $a_1 = 2$

→ Soln.

The recurrence r/n given by, $a_n = 3a_{n-1} - 2a_{n-2}$

Comparing with $a_n = C_1 a_{n-1} + C_2 a_{n-2}$,
$C_1 = 3$, $C_2 = -2$

Characteristic eqn is, $r^2 - C_1 r - C_2 = 0$
or, $r^2 - 3r + 2 = 0$
or, $r^2 - 2r - r + 2 = 0$
or, $r(r-2) - 1(r-2) = 0$
$\therefore r = 1, 2$

Two distinct roots, Theorem 1 applied.

a_n is soln if, $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ —— (i)

For $a_0 = 1$, $a_0 = \alpha_1 1^0 + \alpha_2 2^0$
$1 = \alpha_1 + \alpha_2$ —— (ii)

For $a_1 = 2$, $a_1 = \alpha_1 1^1 + \alpha_2 (2)^1$
$2 = \alpha_1 + 2\alpha_2$ —— (iii)

Solving (ii) and (iii) by elimination method

$$2 = 2\alpha_1 + 2\alpha_2$$
$$-\ 2 = -\alpha_1 + -2\alpha_2$$
$$0 = \alpha_1$$

Hence, $\alpha_1 = 0$

$\alpha_2 = 1$

From (1), $a_n = (0)1^n + (1)(2)^n$

$$\boxed{a_n = 2^n}$$

(20) Represent the following sentences using predicate.
a) Not all heroes are brave
b) Employee who work overtime are awarded.

a) Not all heores are brave

~~let H(x): x is a hero~~

B(x): x is brave.

$\therefore \neg \forall x (H(x) \rightarrow B(x))$

or, $\exists x (H(x) \wedge \neg B(x))$

b) Employee who work overtime are awarded

~~E(x) = x is an employee who works overtime~~

A(x): x is awarded

$\therefore \forall x (E(x) \rightarrow A(x))$

Group D

(21) Define path, Hamilton path and planar graph. Explain any two ways for representing graph with example.

⇒ Path : A sequence of vertices where each adjacent pair is connected by an edge in which neither vertices nor edges are repeated.

Hamilton Path : A path which contain every vertex of a graph G exactly once is called Hamiltoni path.

Euler path : A path which visits every edge of a graph G exactly once.

Two ways of representing a graph.

i) Adjacency matrix

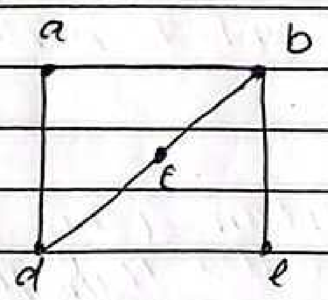
ii) Incident matrix.

Adjacency matrix

⇒ let a_{ij} denote the no. of edges (V_i, V_j) then $A = [a_{ij}]_{m \times m}$ is called adjacency matrix of G if

$$a_{ij} = \begin{cases} 1 & \text{if } (V_i, V_j) \text{ is an edges} \\ 0 & \text{otherwise} \end{cases}$$

eg:

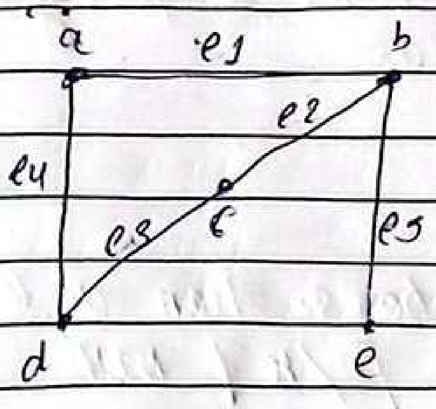


$$A_G = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

Incident Matrix

⇒ Let G be a graph with m vertices $V_1, V_2 \ldots V_m$ and n edges $e_1, e_2, e_3 \ldots e_n$. let a matrix $M = [m_{ij}]_{m \times n}$ defined by.

$$m_{ij} = \begin{cases} 1 & \text{if vertex } V_1 \text{ is incident on } e_j \\ 0 & \text{if } V_1 \text{ is not incident} \\ 2 & \text{if } V_1 \text{ is an end of loop } e_j \end{cases}$$



$$M = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

22a) A company wants to hire a new team of five employee from a pool of 15 candidates. How many different possible teams can be formed if the company requires at least one male and two female. (There are 7 male and 8 female.

Male (7)	Female (8)	Combination
1	4	$C\binom{7}{1} \times \binom{8}{4}$
2	3	$\binom{7}{2}\binom{8}{3}$
3	2	$\binom{7}{3}\binom{8}{2}$

Total combinations $= \binom{7}{1}\binom{8}{4} + \binom{7}{2}\binom{8}{3} + \binom{7}{3}\binom{8}{2}$

$= 7 \cdot 70 + 21 \cdot 56 + 35 \cdot 28$

$= \underline{2646}$

b) A group of 101 people attend a party. Show that there must be at least

Applying pigeonhole principle.

If you put n balls into k boxes, then at least one box contains at least $\lceil n/k \rceil$ balls

Here,
101 people $\rightarrow$ balls
7 days a week $\rightarrow$ boxes

Then $\left\lceil \frac{101}{7} \right\rceil = 15$

$\therefore$ Therefore at least one day of week box contains at least 15 people.

In particular there are at least two people born on same day of week.

Group 'A'

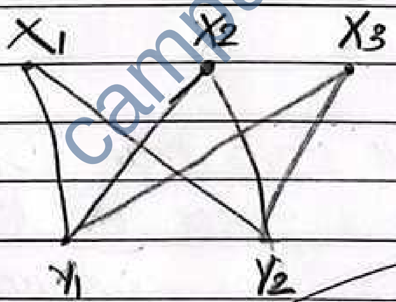
1) Absorption law states that:

$$(p \vee q) \wedge p = p$$

$$(q \vee p) \wedge q = q$$

1

) The graph in which vertex (V) can be partitioned into two subjects X and Y such that each vertex of set X is connected to each vertex of set Y, forms bipartite graph.



1

3

Path: in a graph is traversal / interconnectedness through the vertex along the edges.

Simple path is the Path without loops and multiedges.

Euler Circuit → The circuit formed in which graph traverses through every edges once and only once is called Euler Circuit

Hamilton Circuit → The Circuit formed in which graph traverses through every vertex once and only once except the starting and ending vertex.

1

4

Pigeonhole Principle states that "If k+1 pigeons are to be placed on k boxes, then at least one box contains two or more than two pigeons."

1

Since bit string represent by 0 & 1

$$\underset{(\text{fixed})}{1} \times 7 \times 7 \times 7 \times 7 \times 7 \times 7$$

So possibilities

no. of bit strings of length 7 that begins with one:

$1 \times 7^6 = 117649$ // $1 \times 2^6 = 64$

For the conditional statement $p \rightarrow q$

Converse : $q \rightarrow p$

Contrapositive : $\neg q \rightarrow \neg p$

Inverse : $\neg p \rightarrow \neg q$ example?

0.5

Using pigeonhole principle:

Worst case : 39 cards are choosen of the different suits

Now 3 cards should be choosen which contains same suite — at least 8

Total ways = 39 + 3 = 42 //

$2 \times 4 + 1$

$= 9$

8 Principle of Inclusion and Exclusion

If a task can be done in n_1 ways and same task can also be done in n_2 ways. Then total ways of doing the task is the addn of n_1 and n_2 ways minus the total no. of ways both tasks can be done

$$\therefore P(n_1 \cup n_2) = P(n_1) + P(n_2) - P(n_1 \cap n_2)$$

1

9 The least no. of spanning tree that can be obtained from a connected graph which traverses through every vertex without actually forming circuit or loop is called minimum spanning tree. It is the sub tree of a connected graph.

1

10 For graph $G(V, E)$ where m is the no of edges present, Handshaking theorem states that:

$$2m = \sum_{v=V} deg(v)$$

1

Summation of degrees of each vertex of graph equals to 2 times the edges.

Group B

11) Rule of inference for propositional logic defines set of rules that can be used to generate conclusion on the basis of given premises.
Some of the rules of inference are:

① Modus Ponens

$$\begin{array}{c} p \rightarrow q \\ \underline{\quad p \quad} \\ \therefore q \end{array}$$

eg: If it rains, roads become muddy
It rains
∴ Roads become muddy

② Modus Tollens

$$\begin{array}{c} p \rightarrow q \\ \underline{\neg q} \\ \therefore \neg p \end{array}$$

1. If it rains, roads become muddy
2. Roads didnt become muddy

It didnt rain.

(3) Hypothetical syllogism

$p \to q$
$q \to r$
$\therefore p \to r$

1. If I win, I'll be happy
2. If I'll be happy, I'll dance

$\therefore$ If I win, I'll dance

(4) Disjunctive syllogism

$p \lor q$
$\neg p$
$\therefore q$

1. I like icecream or chocolate
2. I don't like icecream

$\therefore$ I like chocolate

(5) Conjunction

p
q
$\therefore p \land q$

$$\frac{p \wedge q}{p} \qquad \frac{p \vee q}{p}$$

(6) Simplification :-

$$\frac{p \wedge q}{\therefore p}$$

1) I like icecream and chocolate

∴ I like icecream

(7) Resolution

$$\frac{\begin{array}{c} p \vee q \\ \neg p \vee r \end{array}}{q \vee r}$$

1. I prefer tea or coffee
2. I dont prefer tea or I prefer juice

∴ I prefer coffee or juice.

(3)

12

Coefficient of $x^{12}y^{13}$ in the expansion of $(x+y)^{25}$

→ Here,

General term for $(x+y)^{25}$

$$t_{r+1} = C\binom{25}{r} x^{25-r} y^{r} \quad \text{--- (I)}$$

Given, $r = 13$

$$\therefore \quad t_{13+1} = \binom{25}{13} x^{25-13} y^{13}$$

$$= \binom{25}{13} x^{12} y^{13} \quad \text{--- (II)}$$

From (II) we can see that coefficient of $x^{12}y^{13}$ is $C(25, 13)$

$$= \frac{25!}{(25-13)!\,13!}$$

$$= \frac{25!}{12! \times 13!}$$

$$= 5200300$$

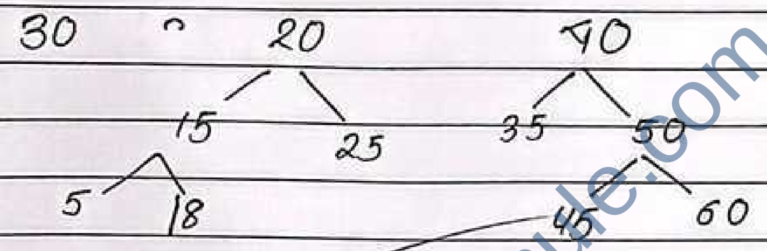
3

13

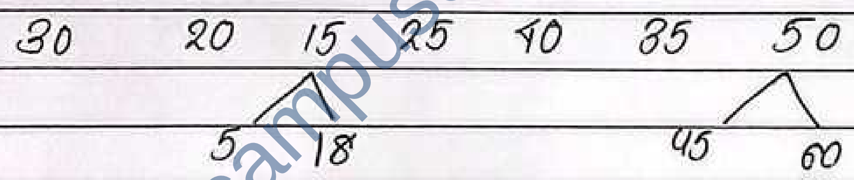
Pre-order traversal (root-left-right)

Traversal which follows root at first and then moves towards left node and finally right node is called pre-order traversal.

Step I



Step II



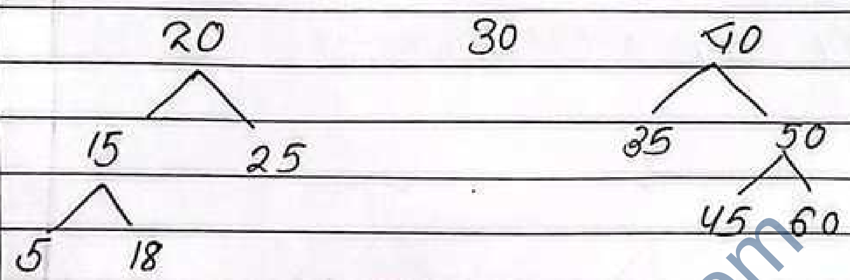
Step III

30 - 20 - 15 - 5 - 18 - 25 - 40 - 35 - 50 - 45 - 60

Inorder Traversal (Left, Root, Right)

Traversal which follows left node at first and then moves towards root node and finally right node.

Step I



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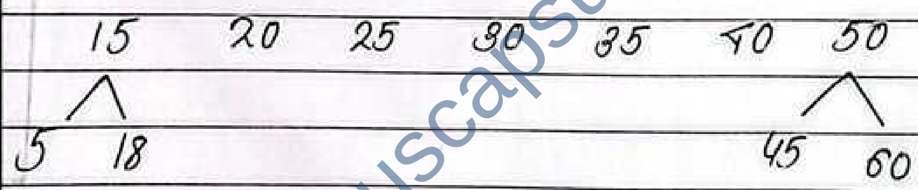


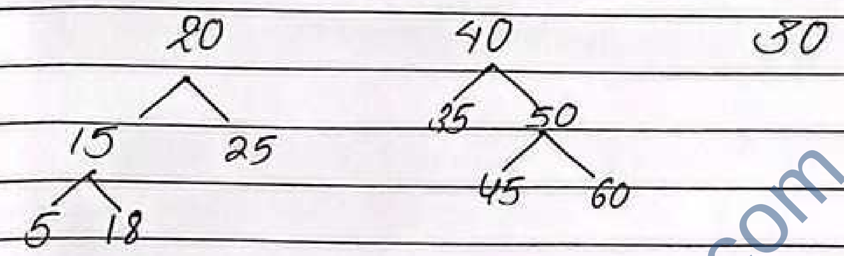
Step III

5 15 18 20 25 30 35 40 45 50 60

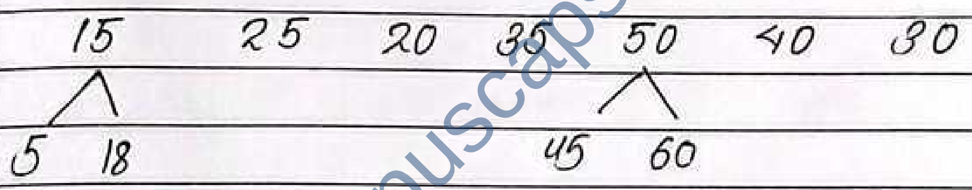
Post order Traversal (Left- Right- Root)

Traversal which follows left node at first an then moves towards right node and finally root node.

Step 1



Step II

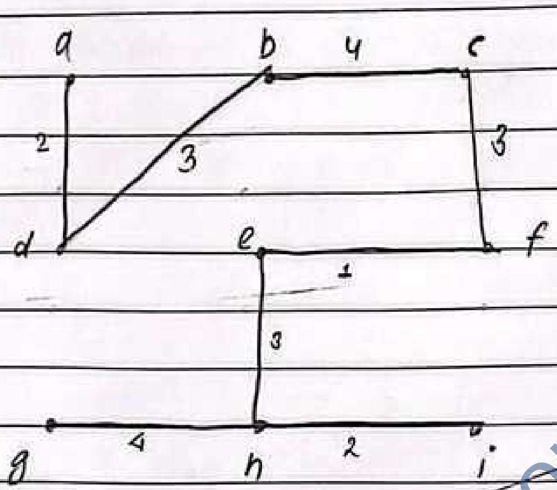


Step III

5 18 15 25 20 35 45 60 50 40 30

(3)

(15) Prim's Algorithm



steps	edges	weights
1	{e, f}	1
2	{e, h}	3
3	{h, i}	2
4	{c, f}	3
5	{b, c}	4
6	{b, d}	3
7	{a, d}	2
8	{g, h}	4
		22

Length of MST = 22

6

Proof by contradiction

If n^2 is even then n is also even integer

Here, $p \rightarrow$. n^2 is even

$q \rightarrow n$ is even integer

To prove by contradiction we assume that $\neg q$ is true i.e.

$\neg q$: n is odd integer

If n is odd integer then there exists an integer k for which $n = 2k+1$

Now,

$$n^2 = (2k+1)^2$$

$$= 4k^2 + 4k + 1$$

$$\cancel{= 2k(2k+}$$

$$= 2(2k^2 + 2k) + 1$$

$$= 2k' + 1 \quad \text{where } 2k^2 + 2k = k'$$

Which is odd, This contradicts the given statement.

Hence by contradiction, we can prove that if n^2 is even then n is also even.

Group C

17)

let us consider following:

$C(x)$: x is in the class
$B(x)$: x studies BIM
$D(x)$: x studies Discrete structure
$L(x)$: x studies Digital logic

Given premises are:

1) $\exists x\,(C(x) \wedge B(x))$
2) $\forall x\,(B(x) \rightarrow (D(x) \wedge L(x)))$

3) leads to the conclusion that

$\therefore \exists x\,(C(x) \wedge D(x))$

Proof

1.	$\exists x(C(x) \wedge B(x))$	Given premises
2.	$C(a) \wedge B(a)$	Existential instantiation, 1
3.	$\forall x\,(B(x) \rightarrow (D(x) \wedge L(x)))$	Given premises
4.	$B(a) \rightarrow (D(a) \wedge L(a))$	Universal instantiation, 3
5.	$B(a)$	Simplification, 2
6.	$D(a) \wedge L(a)$	Modus Ponens, 4 and 5
7.	$C(a)$	Simplification, 2
8.	$D(a)$	Simplification, 6
9)	$C(a) \wedge D(a)$	Conjunction, 7 and 8
10)	$\exists x\,(C(x) \wedge D(x))$	Existential generalization, 9

Hence, proved.

5

18) Given linear congruence $5x \equiv 2 \pmod{26}$ —(i)

Comparing (i) with $ax \equiv b \bmod m$, We get
$a=5$, $b=2$, $m=26$

To find $GCD(26, 5)$

$\therefore \quad 26 = 5 \times 5 + 1$ —(ii)

$5 = 1 \times 5 + 0$

$\therefore GCD(26, 5) = 1$

$1 \mid 2$, Hence solutions can be found

Rewriting eqn (ii),

$26 - 1 = 5 \times 5$

or, $5 \times 5 = 26 - 1$

or, $5 \times (5) \equiv (26-1) \bmod 26$

or, $5 \times 5 \equiv 26 \bmod 26 - 1 \bmod 26$

or, $5(5) \equiv 0 - 1 \bmod 26$

or, $5(-5) \equiv 1 \bmod 26$

or, $5(-5 \times 2) \equiv 2 \bmod 26$

or, $5(-10) \equiv 2 \bmod 26$

Comparing with (i)

$x \equiv -10 \bmod 26$

$= (26-10) \bmod 26$

$= 16 \bmod 26$

$= 16 + 26K$ where $K \in Z$

is the required solution.

(19) Conditions for Isomorphism in graphs are:

(1) no. of vertices of both graphs must be equal.

(2) no. of edges of both graphs must be equal

(3) equal vertices should have same degrees i.e degrees sequence among both graphs must be equal.

(4) vertex correspondance and edge correspondance within graphs must be valid.

Isomorphism Test
In Graph G and H

i) no. of vertex are equal i.e 6

ii) equal no. of edges i.e 8 edges

iii) degrees in G : 2, 3, 3, 2, 3, 3
degrees in H : 2, 3, 3, 2, 3, 3

Hence degrees sequence are also equal.

iv) for vertex correspondance and edge correspondance

Graph G		Graph H
u_1	$\longrightarrow$	v_1
u_2	$\longrightarrow$	v_6
u_3	$\longrightarrow$	v_5
u_4	$\longrightarrow$	v_4
u_5	$\longrightarrow$	v_3
u_6	$\longrightarrow$	v_2

Valid vertex edge correspondance.

Hence graphs G and H are isomorphic.

3

Check the slide since there is one condition provided i.e;

H contain circuit of 3 vertex where as no circuit of 3 vertex is formed in G so no isomorphic

Group D

20) The Chinese Remainder theorem for the system of congurence is given by:

$$x \equiv a_1 \bmod m_1$$
$$x \equiv a_2 \bmod m_2$$
$$x \equiv a_3 \bmod m_3$$
$$\vdots$$
$$x \equiv a_n \bmod m_n$$

Where. a_1, a_2, a_3 — — a_n are constant coefficients and m_1, m_2, m_3 —— m_n must be relatively prime.

x can be calculated by,

$$x = \left(a_1 M_1 M_1^{-1} + a_2 M_2 M_2^{-1} + \ldots a_n M_n M_n^{-1}\right) \bmod M$$

Chinese Remainder Theorem for system of congruence is given by

$$x \equiv a_1 \bmod m_1$$
$$x \equiv a_2 \bmod m_2$$
$$x \equiv a_3 \bmod m_3$$
$$x \equiv a_4 \bmod m_4$$

Given question is,

$$x \equiv 1 \bmod 2$$
$$x \equiv 2 \bmod 3$$
$$x \equiv 3 \bmod 5$$
$$x \equiv 4 \bmod 11$$

Here m_1, m_2, m_3, m_4 are relatively prime. Hence CRT can be calculated.

We know,

$M_n = \frac{M}{m_n}$

$a_1 = 1$	$m_1 = 2$	$M_1 = 165$	$M_1^{-1} = 1$
$a_2 = 2$	$m_2 = 3$	$M_2 = 110$	$M_2^{-1} = 2$
$a_3 = 3$	$m_3 = 5$	$M_3 = 66$	$M_3^{-1} = 1$
$a_4 = 4$	$m_4 = 11$	$M_4 = 30$	$M_4^{-1} = 7$

We know,

$$M = m_1 \times m_2 \times m_3 \times m_4$$
$$= 2 \times 3 \times 5 \times 11$$
$$= 330$$

220

For M_1^{-1}

$$M_1 \times M_1^{-1} = 1 \bmod m_1$$
$$165 \times 1 = 1 \bmod 2$$

120
150
110
210

For M_2^{-1}

$$M_2 \times M_2^{-1} = 1 \bmod m_2$$
$$110 \times 2 = 1 \bmod 3$$

For M_3^{-1}

$$M_3 \times M_3^{-1} = 1 \bmod m_3$$
$$66 \times 1 = 1 \bmod 5$$

For M_4^{-1}

$$M_4 \times M_4^{-1} = 1 \bmod m_4$$
$$30 \times 7 = 1 \bmod 11$$

Now,

$$x = (a_1 M_1 M_1^{-1} + a_2 M_2 M_2^{-1} + a_3 M_3 M_3^{-1} + a_4 M_4 M_4^{-1}) \bmod m$$
$$= (1 \times 165 \times 1 + 2 \times 110 \times 2 + 3 \times 66 \times 1 + 4 \times 30 \times 7) \bmod 330$$
$$= (1643) \bmod 330$$
$$= 323$$

10

(22) Linear Homogeneous recurrence relation for constant coefficients c and degree k is defined by:

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + c_3 a_{n-3} + \cdots c_k a_{n-k}$$

Non Homogeneous recurrence relation for coefficient c and ~~degree k~~ is given by,

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots c_k a_{n-k} + x$$

where x is constant

In non homogeneous recurrence relation each term doesn't have same degree.

→ Given,

$$a_n = -3a_{n-1} - 3a_{n-2} \quad \text{——} (1)$$

Comparing the eqn with standard linear homogeneous recurrence relation,

$$a_n = c_1 a_{n-1} + c_2 a_{n-2}$$

We get,

$$c_1 = -3, \quad c_2 = -3$$

$$-\frac{3}{2}+\frac{\sqrt{3}}{2}i$$

Finding the roots by solving eqn.
We have,

$$r^2 - C_1 r - C_2 = 0$$

or, $r^2 - (-3)r - (-3) = 0$

or, $r^2 + 3r + 3 = 0$

~~or, $r^2 +$~~

~~Comparing the eqn with ax^2+bx+c, we get~~

$a=1$, $b=3$, $c=3$

We have,

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$= \frac{-3 \pm \sqrt{9-12}}{2}$$

$$= \frac{-3 \pm \sqrt{-3}}{2}$$

$$= \frac{-3}{2} \pm \frac{\sqrt{3}}{2} i$$

roots are, $r_1 = -\frac{3}{2} + \frac{\sqrt{3}}{2} i$

$r_2 = -\frac{3}{2} - \frac{\sqrt{3}}{2} i$

Since roots are distinct. Theorem 1 is used.

For each $\{a_n\}$ sequence, theorem 1 is ~~given by~~, true if and only if:

$$a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$$

Substituting value of roots in above eqn,

$$a_n = \alpha_1\left(\frac{-3}{2}+\frac{\sqrt{3}}{2}i\right)^n + \alpha_2\left(\frac{-3}{2}-\frac{\sqrt{3}}{2}i\right)^n \quad \text{--- (ii)}$$

When $a_0 = 1$, $n=0$ from (ii)

$$a_0 = \alpha_1\left(\frac{-3}{2}+\frac{\sqrt{3}}{2}i\right)^0 + \alpha_2\left(\frac{-3}{2}-\frac{\sqrt{3}}{2}i\right)^0$$

$$\boxed{1 = \alpha_1 + \alpha_2} \quad \text{--- (iii)}$$

When $a_1 = -2$, $n=1$ from (ii),

$$a_1 = \alpha_1\left(\frac{-3}{2}+\frac{\sqrt{3}}{2}i\right)^{+1} + \alpha_2\left(\frac{-3}{2}-\frac{\sqrt{3}}{2}i\right)^{+1}$$

$$a_1 = \frac{\alpha_1(-3+\sqrt{3}i)}{2} + \frac{\alpha_2(-3-\sqrt{3}i)}{2}$$

or, $$-2 = \frac{\alpha_1(-3+\sqrt{3}i) + \alpha_2(-3-\sqrt{3}i)}{2}$$

or, $$\boxed{-4 = \alpha_1(-3+\sqrt{3}i) + \alpha_2(-3-\sqrt{3}i)} \quad \text{--- (iv)}$$

Solving (iii) and (iv) by elimination method

$$(-3+\sqrt{3}i) = (-3+\sqrt{3}i)\alpha_1 + (-3+\sqrt{3}i)\alpha_2$$

$$-4 = (-3+\sqrt{3}i)\alpha_1 + (-3-\sqrt{3}i)\alpha_2$$

$$+ \quad - \quad -$$

$$1+\sqrt{3}i = (2\sqrt{3}i)\alpha_2$$

$$\alpha_2 = \frac{1+\sqrt{3}i}{2\sqrt{3}i} = \frac{1}{2} - \frac{\sqrt{3}}{6}i$$

Substituting value of α_2 in (iii)

$$\alpha_1 = 1 - \alpha_2$$

$$= 1 - \frac{1+\sqrt{3}i}{2\sqrt{3}i}$$

$$= \frac{1}{2} + \frac{\sqrt{3}}{6}i$$

Substituting the value of α_1 and α_2 in eqn (ii)

$$a_n = \left(\frac{1}{2}+\frac{\sqrt{3}}{6}i\right)\left(\frac{-3}{2}+\frac{\sqrt{3}}{2}i\right)^n + \left(\frac{1}{2}-\frac{\sqrt{3}}{6}\right)\left(\frac{-3}{2}-\frac{\sqrt{3}}{2}i\right)^n$$

is the required ~~eq~~. solution.

9.5

October 2023.
Discrete Structure

Group "A"

1. Give the negation of the sentence, "All people are loyal."

=) Negation is, There exists at least one person who is not loyal.

2. What is the value of -7 MOD 5?

=) $-7 \div 5$

or, $-7 = 5 \times (-2) + 3$

$\therefore -7 \text{ MOD } 5 = 3$

3. What is recursively defined set?

=) A recursively defined set is a set defined by specifying a few initial steps (base case) and then providing rules (the recursive step) for forming new elements from existing ones.

(4) If the multiple tasks have dependency, can we apply sum rule?

⇒ No, sum rule is used when tasks are mutually exclusive (task cannot occure at same time). If tasks have dependencies, we use product rule.

(5) Differentiate between tree and graph.

⇒ A tree is a connected graph with no cycles. A general graph have cycles and may not be connected. Trees have hierarchial structure with a root node while graphs do not have a designated root.

(6) What is necessary and sufficient condⁿ for a graph to have Euler path but not a Euler circuit?

⇒ The condition is that the graph must be connected and has exactly two vertices of odd degree.

7) Define prefix code.

⇒ Prefix code is the computer friendly code in which operator precedes its operands.

eg: + ab

−+ abc

8) What is spanning tree?

⇒ A spanning tree of a connected, undirected graph is a subgraph that is a tree and includes all the vertices of the original graph.

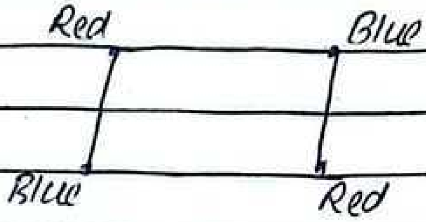
9) What do you mean by well ordering property?

⇒ The well-ordering property states that every non-empty set of non-negative integers has a least element. This means that if you have a group of whole numbers, there'll be one number that is smallest among them.

eg: in the set {3, 5, 1, 4}.
the least element is 1.

(10) What is chromatic number for planar graph?

⇒ The least number of colors required for coloring of a graph G is called chromatic number.



$\chi(G) = 2$

Group "B"

(11) Prove that $\sqrt{2}$ is irrational number.

→ Soln,

let $\sqrt{2}$ is not irrational number.

According to defn of rationality,

$\sqrt{2} = \frac{a}{b}$ where a, b are in lowest forms & $b \neq 0$

i.e $\sqrt{2} = \frac{a}{b}$

or, $2 = \frac{a^2}{b^2}$

or, $2b^2 = a^2$ —— (i)

Here a^2 is even.

Since a^2 is even, a is also even.

By def^n of even, $a = 2k$ where k is some integer

Substituting for a in eq^n (i)

$$2b^2 = (2k)^2$$

or, $2b^2 = 4k^2$

or, $b^2 = 2k^2$

Here b^2 is even

Since b^2 is even, b is even.

By def^n of even,

$b = 2m$ —— (ii) where m is some integers

$$\sqrt{2} = \frac{a}{b} = \frac{2k}{2m}$$

Where $\frac{2k}{2m}$ is not in there lowest form. which contradi our supposition. (Thus $\sqrt{2}$ is irrational)

Date ______
Page ______

(12) Use mathematical induction to prove $n^5 - n$ is divisible by 5.

→ Soln.

(1) Base step: Here, $P(1)$ is true because $1^5 - 1 = 0$ which is divisible by 5. Hence basis step is proved.

(2) Inductive step: Assume $P(k)$ is true i.e $k^5 - k$ is divisible by 5.

Thus using $P(k)$ we have to show that $P(k+1)$ is true.

i.e $(k+1)^5 - (k+1)$ is divisible by 5

Now, solving for $P(k+1)$

$$= (k+1)^5 - k+1$$

$$= k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1 - k - 1$$

$$= (k^5 - k) + (5k^4 + 10k^3 + 10k^2 + 5k)$$

$$= (k^5 - k) + 5(k^4 + k^3 + 2k^2 + k)$$

Here $(k^5 - k)$ is divisible by 5 and the term $5(k^4 + k^3 + 2k^2 + k)$ is multiple of 5, which is divisible by 5.

Thus $P(k+1)$ is divisible by 5.

(13) Define the binomial coefficient. Prove that $C(n+1, r) = C(n, r-1) + C(n, r)$

Binomial coefficient $\binom{n}{k}$ is the number of ways of picking k unordered outcomes from n possibilities, also known as combination.

To prove,

$$C(n+1, r) = C(n, r-1) + C(n, r)$$

or, $$\frac{(n+1)!}{(n+1-r)!\,r!} = C(n, r-1) + C(n, r)$$

Taking RHS.

$$\frac{n!}{(n-r+1)!\,(r-1)!} + \frac{n!}{(n-r)!\,r!}$$

$$= \frac{n!}{(n-r+1)(n-r)!\,(r-1)!} + \frac{n!}{(n-r)!\,r(r-1)!}$$

$$= \frac{n!}{(n-r)!\,(r-1)!}\left[\frac{1}{n-r+1} + \frac{1}{r}\right]$$

$$= \frac{n!}{(n-r)!\,(r-1)!}\left[\frac{r+n-r+1}{r(n-r+1)}\right]$$

$$= \frac{n!}{(n-r)!\,(r-1)!}\left[\frac{n+1}{n(n-r+1)}\right]$$

$$= \frac{(n+1)!}{(n-r+1)!\,r!}$$

$$= \frac{(n+1)!}{(n+1-r)!\,r!} = LHS$$

Hence proved //

$s = s_1 - q s_2$
$t = t_1 - q t_2$

(14) Find GCD of 20 and 24 using Extended Euclidean algorithm.

→ Solution, GCD(24, 20)

q	a	b	r	s_1	s_2	s	t_1	t_2	t
1	24	20	4	1	0	1	0	1	-1
5	20	4	0	0	1	-5	1	-1	6
	$\boxed{4}$	0		$\boxed{1}$	-5		$\boxed{-1}$	6	

Initially, $s_1 = 1$ $s_2 = 0$ | $s = s_1 - q s_2$
$t_1 = 0$ $t_2 = 1$ | $t = t_1 - q t_2$

We have,

$$GCD = 4, \quad s = 1, \quad t = -1$$

Also, $s \cdot a + t \cdot b = gcd(a, b)$

$1 \cdot 24 + (-1) \cdot 20 = 4$

$24 - 20 = 4$

$4 = 4$

Hence proved //

(15) What are the necessary conditions for a graph to be isomorphic.

⇒ Two graphs G_1 and G_2 are isomorphic if:

(i) no. of vertices in G_1 = no. of vertices in G_2

(ii) no. of edges in G_1 = no. of edges in G_2

(iii) Degree sequence in G_1 = Degree sequence in G_2

(iv) $\deg(u) = K$ in G_1, then $\deg(\phi(u)) = K$ in G_2

(v) $\{u, v\} \in E(G_1) \longrightarrow \{\phi(u), \phi(v)\} \in E(G_2)$

(vi) $A(G_1) = A(G_2)$ w.r.t vertices in ϕ
where. A = adjacency matrix

Group "C"

(17) Test the validity of an argument: Someone in this class has studied BIM. Everyone who has studied BIM study DS and DL. Therefore someone in this class has studied DS.

let the predicates be:

$B(x)$: x studied BIM

$D(x)$: x studied DS

$L(x)$: x studied DL

where domain of discourse is x is the student in this class.

Premises are:

(1) $\exists x\, B(x)$

(2) $\forall x\,(B(x) \rightarrow (D(x) \wedge L(x)))$

Conclusion: $\exists x\, D(x)$

To prove conclusion using premises and rule of inferences

Proof:

1	$\exists x\, B(x)$	Premise no. 1
2	$B(a)$	Existential instantiation of 1
3	$\forall x\,(B(x) \rightarrow D(x) \wedge L(x)))$	Premise no. 2
4	$B(a) \rightarrow D(a) \wedge L(a)$	Universal instantiation of 3
5	$D(a) \wedge L(a)$	Modus ponens using 2 and 4
6	$D(a)$	Simplification using 5
7	$\exists x\, D(x)$	Existential generalization of 6

Hence, the argument is valid.

(18) Using CRT solve the following congruence

$x \equiv 1 \pmod 2$, $x \equiv 4 \pmod 5$, $x \equiv 2 \pmod 7$

⇒ Soln.

The Chinese remainder theorem is of the form.

$x \equiv a_1 \bmod m_1$

$x \equiv a_2 \bmod m_2$

$x \equiv a_3 \bmod m_3$

Where m_1, m_2, m_3 are relatively prime.

From the given question we have.

$x \equiv 1 \bmod 2$

$x \equiv 4 \bmod 5$

$x \equiv 2 \bmod 7$

Here 2, 5, 7 are relatively prime. Hence CRT can be performed.

$$M_n = \frac{M}{m_1}$$

$a_1 = 1$	$m_1 = 2$	$M_1 = 35$	$M_1^{-1} = 1$	
$a_2 = 4$	$m_2 = 5$	$M_2 = 14$	$M_2^{-1} = 4$	$M = 70$
$a_3 = 2$	$m_3 = 7$	$M_3 = 10$	$M_3^{-1} = 5$	

$M = m_1 \cdot m_2 \cdot m_3$

$= 2 \times 5 \times 7 = 70$

For M_1^{-1}

$$M_1 \cdot M_1^{-1} = 1 \bmod m_1$$
$$35 \cdot 1 = 1 \bmod 2$$

For M_2^{-1}

$$M_2 \cdot M_2^{-1} = 1 \bmod m_2$$
$$14 \cdot 4 = 1 \bmod 5$$

For M_3^{-1}

$$M_3 \cdot M_3^{-1} = 1 \bmod m_3$$
$$10 \cdot 5 = 1 \bmod 7$$

Substituting the values in formula:

$$x = (a_1 M_1 M_1^{-1} + a_2 M_2 M_2^{-1} + a_3 M_3 M_3^{-1}) \bmod M$$

$$= (1 \times 35 \times 1 + 4 \cdot 14 \cdot 4 + 2 \cdot 10 \cdot 5) \bmod 70$$

$$= 359 \bmod 70$$

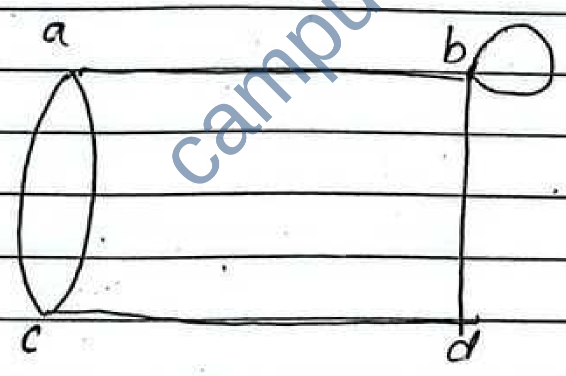
$$= 9 \bmod 70$$

$$\therefore x = 9 + 70k, \ k \in \mathbb{Z}$$

(19) How do you compute the degree of a node in directed and undirected graphs? Explain with an example.

In an undirected graph

- The degree of vertex is the number of edges ~~to~~ incident to it
- Eeach edga connected to vertex contributes 1 to its degree
- Loops contribute 2 to its degree.



d(a) = 3
d(b) = 3
d(c) = 3
d(d) = 2

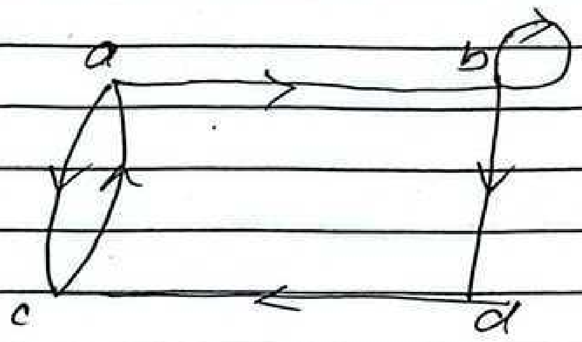
In an directed graph

- Each vertex has:

  Indegree = no. of edges coming into the vertex

  Outdegree = no. of edges going out of the vertex.

- A loop contribute 1 to indegree and outdegree



Indegree	Outdegree
$d^{-1}(a) = 1$	$d^{-1}(a) = 2$
$d^{-1}(b) = 2$	$d^{-1}(b) = 2$
$d^{-1}(c) = 2$	$d^{-1}(c) = 1$
$d^{-1}(d) = 1.$	$d^{-1}(d) = 1$

Group "D"

(21) Define linear homogeneous recurrence relation. Solve,

$a_n = a_{n-1} + a_{n-2}$, $n \geq 3$ with initial condn $a_1 = 1$ and $a_2 = 1$.

⇒ The linear homogeneous recurrence relation of degree k with constant coefficients is a recurrence r/n of the form.

$a_n = C_1 a_{n-1} + C_2 a_{n-2} + \text{——} + C_k a_{n-k}$.

where $C_1, C_2, \text{——} C_k$ are real no.
$C_k \neq 0$

Given eqn.

$a_n = a_{n-1} + a_{n-2}$, $a_1 = 1$ $a_2 = 1$ —— (*)

eqn (1) is in the form $C_1 a_{n-1} + C_2 a_{n-2}$
where $C_1 = 1$, $C_2 = 1$

Characteristics eqn is. $r^2 - C_1 r - C_2 = 0$
or, $r^2 - r - 1 = 0$

Comparing the eqn with $ax^2 + bx + c$ where $x = r$.
We get, $a = 1$, $b = -1$, $c = -1$

We have,

$$r = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$= \frac{+1 \pm \sqrt{(-1)^2 - 4\cdot 1\cdot(-1)}}{2\cdot 1}$$

$$= \frac{1 \pm \sqrt{1+4}}{2}$$

$$= \frac{1 \pm \sqrt{5}}{2}$$

Hence,

$$r_1 = \frac{1+\sqrt{5}}{2}, \quad r_2 = \frac{1-\sqrt{5}}{2}$$

Since there are two distinct roots, theorem 1 is applied.

i.e a_n is soln iff, $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$

$$\boxed{a_n = \alpha_1\left(\frac{1+\sqrt{5}}{2}\right)^n + \alpha_2\left(\frac{1-\sqrt{5}}{2}\right)^n} \text{ — (i)}$$

For $a_1 = 1$, $a_1 = \alpha_1\left(\frac{1+\sqrt{5}}{2}\right)^1 + \alpha_2\left(\frac{1-\sqrt{5}}{2}\right)^1$

$$\text{or } 1 = \left(\frac{1+\sqrt{5}}{2}\right)\alpha_1 + \left(\frac{1-\sqrt{5}}{2}\right)\alpha_2$$

or, $2 = (1+\sqrt{5})\alpha_1 + (1-\sqrt{5})\alpha_2$ — (ii)

For $a_2 = 1$.

$$a_2 = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right)^2 + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)^2$$

or, $1 = \alpha_1 \left(\frac{3+\sqrt{5}}{2}\right) + \alpha_2 \left(\frac{3-\sqrt{5}}{2}\right)$

or, $2 = \alpha_1 (3+\sqrt{5}) + (3-\sqrt{5})\alpha_2$ ——— (iii)

Substituting value of α_1 and α_2 using elimination method of (ii) and (iii)

$$(1+\sqrt{5}) = \cancel{(3+\sqrt{5})}\alpha_1 - 2\alpha_2$$

$$2 = \cancel{(3+\sqrt{5})}\alpha_1 + (3+\sqrt{5})\alpha_2$$

$$(-) \qquad (-) \qquad (-)$$

$$-1+\sqrt{5} = (-5+\sqrt{5})\alpha_2$$

$$\alpha_2 = \frac{-\sqrt{5}}{5} = \frac{-1}{\sqrt{5}}$$

Substituting value of α_2 in eqn (ii)

$$2 = (1+\sqrt{5})\alpha_1 + (1-\sqrt{5})\cdot\left(\frac{-1}{\sqrt{5}}\right)$$

$$2 = (1+\sqrt{5})\alpha_1 + \frac{5-\sqrt{5}}{5}$$

or, $\alpha_1 = \frac{1}{\sqrt{5}}$

$\therefore$ $\alpha_1 = \frac{1}{\sqrt{5}}$, $\alpha_2 = -\frac{1}{\sqrt{5}}$

Hence sol^n is.

$$a_n = \left(\frac{1+\sqrt{5}}{2}\right)^n \frac{1}{\sqrt{5}} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$ //

2)

Q. A box contain 2 white balls, 3 black balls and 4 red balls. In how many ways 3 balls can be drawn from the box if at least one black ball is to be included in the draw.

Soln. Total balls = 2+3+4 = 9

Total ways to choose 3 from 9 = $\binom{9}{3}$

$= 84$

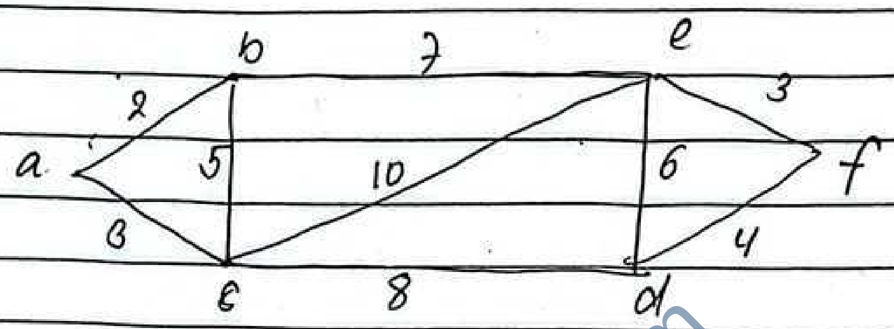
When no black ball is choosen, we choose all 3 from 2 white + 4 red = 6 balls

i.e $\binom{6}{3} = 20$

Ways with at least one black ball = 84 − 20

= 64 ways //

(b) Apply Dijkstra's algorithm to find the shortest path between a to f



a	b	c	d	e	f
0	∞	∞	∞	∞	∞
0	2	7	∞	9	∞
0	2	7	15	9	∞
0	2	7	15	9	12
0	2	7	15	9	12
0	2	7	15	9	12

Shortest path between a-f is 12

trail is : a - b - e - f